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T. G. Stryzhak

THE NEW EXTREMAL CRITERION OF STABILITY

Introduction

It would be rather difficult to list all the publications about the Pendulum. Throughout history, oscillations of pendulum systems have been researched by such illustrious scientists as Galileo, Newton, Euler, Huygens, as well as many others. Their contributions into studying the mechanisms of pendulum motion were of great importance in the history of mankind's discoveries and technological progress. The pendulum has been used in clocks to define time, in devices for measuring terrestrial gravitation, as a plumb line in building to define the vertical, etc. The Earth's rotation was proved with the help of the pendulum oscillations as well.

Pendulum systems are, as a rule, non–linear and require specific methods of research. For instance, creation of the elliptical function theory by Abel, Jacoby, and Weierstrass is also connected with the research of the mathematical pendulum oscillations.

Nowadays due to the growing amount of mathematics involved in the research of different sciences there has been an increasing interest in studying motions of pendulum systems. The following special terms have appeared: the pendulum law of the population migration, the pendulum law of the rhythm regulator action, the pendulum of emotions, etc.

It is a well–known fact that all living organisms have so called biological clocks, at the basis of which there is an oscillating system – a non–linear oscillator. It is well known that the vestibular apparatus of animals and humans contains three non–linear pendulum systems, which are located in three mutually perpendicular planes. Oscillations are present everywhere: in the opening of a flower at the sunrise, in the growth of an embryo, in germinating of a grain, in the heart beating, in the work of jackhammers and dental drills, in the rise and fall of the tide. And wherever there is life, there are oscillations at the cellular level.

The nature of forces causing the oscillations is variable, but the result is the same – oscillations.

This information bears a descriptive character as we would like to attract the readers' attention to the oscillation theory to convince them of the fact that oscillation processes imply something much deeper than what we actually know: that the source of the oscillating processes is a hidden potential force and kinetic energy.

Oscillations of a non–linear oscillator can be forced or can have a free–running character (such as the heart or aorta beating, biorhythms, etc.). In fact, the pendulum motion laws, periodic or almost–periodic oscillations, are immanent in the whole physical world.

A human being receives the main information about the outer world via sound and light oscillations. Pendulum systems are used to analyse various engineering tasks. For example, oscillations in electrical and mechanical systems, rocking of ships on water, oscillations of satellites, vibration of the hull and the wings of planes, movements of cables, chains, travelling or gantry cranes, etc. All these phenomena are explained with analogous differential equations which describe motions.

At this point our introduction is completed. In this book we shall use the beautiful algorithmic mathematical language: ***"A" is given, "B" is to be proved***.

Problem Formulation

The new extremal criterion of stability

The stability of oscillations of a mechanical or electrical system, which is affected by low amplitude high–frequency disturbances, is researched. It is shown that the effect of vibration can be replaced by the action of some potential force. Under the action of some disturbance a stable equilibrium position can become unstable, and an unstable equilibrium position can become stable. New positions of stable dynamic equilibrium can also appear. Conclusions about the stability can be made with the help of the new extremal criterion for stability described in this work. The criterion for stability rests on the time averaging operation of the canonical system of differential equations. The new extremal criterion of stability is applied when the frequency of external disturbance is substantially higher than the natural vibration of the quiescent system.

In this work the new extremal criteria of stability is applied in the research of the stability of the pendulum systems equilibrium position. In particular, it is found that any position of the pendulum, even a horizontal one, can be made stable via vibration of the suspension point of a pendulum.

An experimental installation to check the received theoretical conclusions was created. All the conclusions were proved correct.

This work illustrates simple examples of the extremal criteria for stability application. It also describes the experiment. At the end of this work the mathematical survey of the extremal criteria of stability is presented.

The new extremal stability criterion definition

Here the movement of a mechanical or electrical system is studied. We shall define the generalized coordinates by q_j $(j=1,...,n)$, the generalized velocities by $\dot{q}_j$ $(j=1,...,n)$ and the time by t.

Let $L=L(t,\dot{q}_j,q_j)$ be the kinetic potential, where $L=\mathrm{T}-\Pi$, and T is the kinetic energy of the system and Π is the potential energy. We shall find the minimum of the kinetic potential L as a function of the generalized velocities $\dot{q}_j$,

$$\Pi_0(t,q_j)=\min_{\dot{q}_s} L(t,\dot{q}_j,q_j) \quad , \quad (j,s=1,...,n). \tag{1}$$

We shall exclude from $L(t,\dot{q}_j,q_j)$ the derivative $\dot{q}_j$ by the necessary conditions of extremum,

$$\frac{\partial L(t,\dot{q}_j,q_j)}{\partial \dot{q}_s}=0 \qquad (j,s=1,...,n). \tag{2}$$

Using the operation of time averaging t on the function $\Pi_0(t,q_j)$ we are able to eliminate *t*,

$$\Pi_0(q_j)\equiv\langle\Pi_0(t,q_j)\rangle\equiv\lim_{\mathrm{T}\to+\infty}\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\Pi_0(t,q_j)\,dt. \tag{3}$$

If the function $\Pi_0(q_j)$ has a maximum at the point $q_j=q_{j0}$ $(j=1,...,n)$, then the maximum corresponds to the stable position of the dynamic equilibrium of the initial oscillating system. Meanwhile the system is steadily oscillating at the position $q_j=q_{j0}$ $(j=1,...,n)$.

In order to find the position of the stable state, it is first necessary to find the minimum of the function $L(t,\dot{q}_j,q_j)$ via variable $\dot{q}_j$, and then we find the maximum of the function using

$$\max_{q_j} < \min_{\dot{q}_j} L(t,\dot{q}_j,q_j) >$$

by which the name ***'extremal criterion of stability'*** arises.

It is important to mention that the conditions of stability are found without using Lagrange's differential equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_S}-\frac{\partial L}{\partial q_S}=0 \qquad (s=1,...,n)$$

and only the knowledge of Lagrange's function is used

$$L = L\left(t, \dot{q}_j, q_j\right)$$

Stability of the mathematical pendulum oscillations at the upper equilibrium position

We shall consider oscillations at the upper equilibrium position of the mathematical pendulum with length l and mass m, with the vibrating vertical point of support A. We shall define the coordinates of the centre of gravity of the pendulum by x, y, the pendulum angle from the vertical by φ and the deflection of the point A of the pendulum support in a vertical direction (figure 1) by $r(t) = a \sin \omega t$, where a is the amplitude and $\boldsymbol{\omega}$ is the frequency of the vibrations.

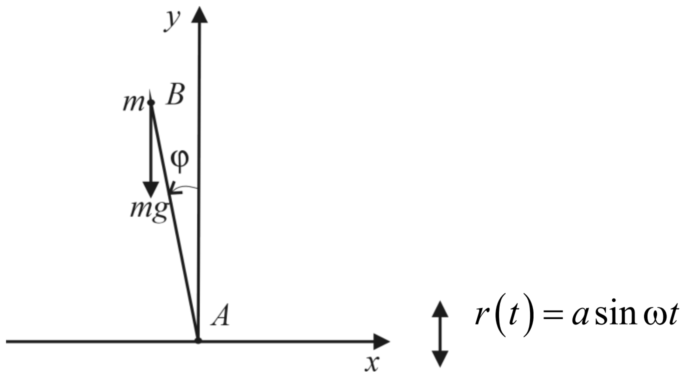

Fig. 1. Stability of the pendulum at the upper equilibrium position

For the coordinates of the centre of gravity we find the following,

$$x = -l \sin \varphi, \; y = l \cos \varphi + r,$$

and for the velocities,

$$\dot{x} = -l \cos \varphi \cdot \dot{\varphi}, \; \dot{y} = -l \sin \varphi \cdot \dot{\varphi} + \dot{r}.$$

The kinetic and potential energies are found to have the following forms,

$$\mathrm{T} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\left(l^2 \dot{\varphi}^2 - 2l\dot{\varphi}\sin\varphi \cdot \dot{r} + \dot{r}^2\right),$$

$$\Pi = mg\left(l \cos \phi + r\right)$$

Then the Lagrange function is as follows,

$$L = \mathrm{T} - \Pi = \frac{m}{2}\left(\left(l\dot{\varphi} - \dot{r}\sin\varphi\right)^2 + \dot{r}^2 \cos^2 \varphi\right) - mg\left(l \cos \varphi + r\right) \qquad (4)$$

and $\min\limits_{\dot{\varphi}} L$ is found with the help of the equation

$$\frac{\partial L}{\partial \dot{\varphi}} = m\left(l\dot{\varphi} - \dot{r}\sin\varphi\right)\cdot l = 0,$$

giving the following expression

$$\min_{\dot{\varphi}} L = \frac{m}{2}\dot{r}^2\cos^2\varphi - mgl\cos\varphi - mgr \tag{5}$$

As $r(t) = a\sin\omega t$ we get

$$\langle r\rangle = \langle a\sin\omega t\rangle = 0, \quad \langle \dot{r}^2\rangle = \langle a^2\omega^2\cos^2\omega t\rangle = \frac{a^2\omega^2}{2}$$

where <X> is the time average of the function X.

Therefore the following is found,

$$\Pi_0(\varphi) = \left\langle \min_{\dot{\varphi}} L \right\rangle = mgl\cos\varphi + \frac{m}{2}\frac{a^2\omega^2}{2}\cos^2\varphi,$$

where $-\Pi_0(\varphi)$ is a dynamic analogue of potential energy.

The function $-\Pi_0(\varphi)$ has its maximum at the point $\varphi = 0$, if the following condition is satisfied:

$$\frac{\partial^2\Pi_0(\varphi)}{\partial\varphi^2} < 0 \quad \text{or} \quad mgl - \frac{m}{2}a^2\omega^2 < 0.$$

The stability criterion is reduced to the following inequality:

$$a^2\omega^2 > 2gl. \tag{6}$$

Relation (6) was discovered earlier in relevant scientific work. It is obvious that it was first discovered in works [1, 2], and later repeated in the works of N. Bogolyubov, P. Kapitsa, G. Stoker, K. Valeev, T. Stryzhak.

Conclusions

“The new extremal criterion of stability” is a criterion sufficient for determining the motion stability of a mechanical system under the influence of small–amplitude high–frequency parameter oscillations. This criterion allows the calculation of stability conditions of the oscillations of a non–stable mechanical system. It does not involve the use of motion equations, rather, it uses only the Lagrange’s function $L = T - \Pi$.

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